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GRADED RINGS IN WHICH EVERY PROPER GRADED IDEAL IS ALMOST GR-PRIME

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Abstract

In this paper, we study further properties of almost and n-almost gr-prime ideals in a graded ring R. In particular, we investigate some conditions under which a graded ideal is almost gr-prime. Finally, we give a characterization for graded rings in which every proper graded ideal is almost gr-prime.

1. Introduction

Throughout our work on almost graded prime ideals which has yet to appear, we develop the theory of almost graded prime ideals and n-almost

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graded prime ideals, and, then we extend some basic results about almost and n-almost prime ideals to the graded case. In this paper we study further properties of almost and n-almost graded prime ideals in a graded ring R.

Let R be a commutative ring and let G be an abelian group. Then R is called a G-graded ring if there exists a family $\{R_g : g \in G\}$ of additive subgroups of R such that $R = \oplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for each g and h in G. Let $h(R) = \bigcup_{g \in G} R_g$. Then any element of R belongs to $h(R)$ is called homogeneous.. Moreover, if $x \in R_g$ for some $g \in G$, then we say that x is of degree g. An ideal I of a graded ring R is called graded if $I = \oplus_{g \in G}(I \cap R_g) = \oplus_{g \in G} I_g$. Equivalently, I is graded in R if and only if I has a homogeneous set of generators. If R and R' are two G-graded rings, then a mapping $\eta : R \to R'$ with $\eta(1_R) = 1_{R'}$ is called a gr-homomorphism if it is a ring homomorphism such that $\eta(R_g) \subseteq R'_g$ for all $g \in G$. Let $R = \oplus_{g \in G} R_g$ be a G-graded ring and let I be a graded ideal of R. Then the Quotient ring R / I is also a G-graded ring. Indeed, $R / I = \oplus_{g \in G}(R / I)_g$, where $(R / I)_g = \{x + I : x \in R_g\}$. Also, if R_1 and R_2 are G_1- graded and G_2- graded rings respictively, then $R_1 \times R_2$ is $G_1 \times G_2$- graded with

$$(R_1 \times R_2)_{(g, h)} = \{(a_g, b_h) : a_g \in R_{1g} \text{ and } b_h \in R_{2h}\},$$

for $(g, h) \in G_1 \times G_2$. Moreover, a G-graded ring R is called gr-decomposable if $R = R_1 \times R_2$ for some non trivial G-graded rings R_1 and R_2. Otherwise, R is called gr-indecomposable..

Let R be a graded ring and let $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fractions $S^{-1}R$ is a graded ring which is called the gr-ring of fractions. Indeed, $S^{-1}R = \oplus_{g \in G}(S^{-1}R)_g$, where

$$(S^{-1}R)_g = \{\frac{r}{s} : r \in h(R),\ s \in S \text{ and } g = (\text{degs})^{-1}(\text{degr})\}.$$

Let $\eta : R \to S^{-1}R$ be a ring gr-homomorphism defined by $\eta(r) = \frac{r}{1}$. Then for any graded ideal I of R, the graded ideal of $S^{-1}R$ generated by $\eta(I)$ is denoted by $S^{-1}I$. Similar to non graded case one can prove that

$$S^{-1}I = \{\lambda \in S^{-1}R : \lambda = \frac{r}{s} \text{ for } r \in I \text{ and } s \in S\},$$

and that $S^{-1}I \neq S^{-1}R$ if and only if $S \cap I = \phi$.

If $\mathcal{J}$ is a graded ideal in $S^{-1}R$, then $\mathcal{J} \cap R$ will denotes the graded ideal $\eta^{-1}(\mathcal{J})$ of R. Moreover, one can prove that

$$S^{-1}I \cap R = \{x \in h(R) : xs \in I \text{ for some } s \in S\}.$$

A graded ideal P of a graded ring R is called gr-prime if whenever $x, y \in h(R)$ with $xy \in P$, then $x \in P$ or $y \in P$. If a graded ideal M of R is maximal in the lattice of graded ideals of R, then M is called gr-maximal in R and the set of all gr-maximal ideals of R is denoted by $J^g(R)$. By using Zorn's Lemma, one can see that if R is a non trivial graded ring, then it contains at least one gr-maximal ideal. A graded ring with unique gr-maximal ideal is called a gr-local ring. Following [1], a graded ideal P of a graded ring R is called weakly gr-prime if whenever $x, y \in h(R)$ with $xy \in P - \{0\}$, then $x \in P$ or $y \in P$. As a generalization of weakly gr-prime ideals, almost gr-prime ideals and n-almost gr-prime ideals have been defined in [3] where $n \in \mathbb{N}$. Indeed, a graded ideal P of a graded ring R is called n-almost gr-prime if whenever $xy \in P - P^n$, then $x \in P$ or $y \in P$, where $x, y \in h(R)$. In particular, the almost gr-prime ideals are just the 2-almost gr-prime ideals.

Let P be any gr-prime ideal of a graded ring R and consider the multiplicatively closed subset $S = h(R) - P$. We denote the graded ring of fraction $S^{-1}R$ of R by R_P^g and we call it the gr-localization of R. This ring is gr-local with the unique gr-maximal $S^{-1}P$ which will be denoted by PR_P^g. Moreover, for graded ideals I and J of R, if $IR_P^g = JR_P^g$ for every gr-prime (gr-maximal) ideal P of R, then $I = J$.

2. Properties of Almost GR-Prime Ideals

In the following Proposition, we can determine two properties of almost gr-prime ideals.

Proposition 2.1. *Let P be an almost gr-prime ideal of a graded ring R. Then*

1. *If I is a graded ideal of R with $I \subseteq P$, then P / I is an almost gr-prime ideal of R / I.*

2. *If $S \subseteq h(R)$ is a multiplicatively closed subset of R with $P \cap S = \phi$, then $S^{-1}P$ is an almost gr-prime ideal of $S^{-1}R$.*

Proof.1. Let $r_1 + I$ and $r_2 + I$ be two elements in $h(R / I)$ such that $(r_1 + I)(r_2 + I) \in P / I - (P / I)^2$. Then $r_1, r_2 \in h(R)$ with $r_1r_2 + I \in P / I - (P^2 + I) / I$. So, $r_1r_2 \in P$ and $r_1r_2 \notin P^2 + I$ and, then $r_1r_2 \notin P^2$. Since P is almost gr-prime, then $r_1 \in P$ or $r_2 \in P$ and therefore, $r_1 + I \in P / I$ or $r_2 + I \in P / I$. Thus, P / I is an almost gr-prime ideal.

2. Let $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in h(S^{-1}R)$ with $\frac{r_1}{s_1} . \frac{r_2}{s_2} \in S^{-1}P - (S^{-1}P^2)$. Then there exist $b \in P$ and $s \in S$ such that $\frac{r_1r_2}{s_1s_2} = \frac{b}{s}$ and, then there exists $u \in S$ such that $usr_1r_2 = us_1s_2b \in P$. Moreover, $wr_1r_2 \notin P^2$ for any $w \in S$ and so $usr_1r_2 \in P - P^2$. Since P is almost gr-prime, then either $usr_1 \in P$ or $r_2 \in P$. Consequently, either $\frac{r_1}{s_1} \in S^{-1}P$ or $\frac{r_2}{s_2} \in S^{-1}P$ which implies that $S^{-1}P$ is almost gr-prime.

Remark 2.2. By the previous proposition, if P is an almost gr-prime ideal of R, then $S^{-1}P$ is an almost gr-prime ideal of $S^{-1}R$. However, $S^{-1}P \cap R \neq P$ in general. For example, let $R = Z_6 \oplus \sqrt{5}Z_6 = R_{\overline{0}} + R_{\overline{1}}$, then R is Z_2-graded ring with $1 \in R_{\overline{0}}$ and let $P = \{0\}$. Then P is an almost gr-prime ideal of R and so $S^{-1}P = S^{-1}\{0\}$ is almost gr-prime in $S^{-1}R$, where $S = \{\overline{1}, \overline{2}, \overline{4}, \sqrt{5}, \overline{2}\sqrt{5}, \overline{4}\sqrt{5}\}$ is a multiplicatively closed subset of R. However,

$$S^{-1}P \cap R = \{x \in R : xs = \overline{0} \textit{ for some } s \in S\} = \{\overline{0}, \overline{3}, \overline{3}\sqrt{5}\} \neq P.$$

Recall that if A and B are two sets, then $(A \times B) - (A \times B)^2 = (A - A^2) \times B$.

Proposition 2.3. *Let R_1 and R_2 be two graded rings. A graded ideal A of $R_1 \times R_2$ is almost gr-prime if and only if A has one of the following forms*:

1. $A = P_1 \times R_2$, *where P_1 is an almost gr-prime ideal in R_1*.

2. $A = R_1 \times P_2$, *where P_2 is an almost gr-prime ideal in R_2*.

3. $A = I \times J$, *where I and J are both idempotent graded ideals in R_1 and R_2 respectively.*

Proof.

$\Rightarrow$) : Suppose that A is almost gr-prime in $R_1 \times R_2$. Let $A = P_1 \times R_2$ for some graded ideal P_1 of R_1. Let $a, b \in h(R_1)$ such that $ab \in P_1 - P_1^2$. Then

$$(a, 1)(b, 1) \in (P_1 - P_1^2) \times R_2 = (P_1 \times R_2) - (P_1 \times R_2)^2.$$

Since $P_1 \times R_2$ is almost gr-prime, then $(a, 1) \in P_1 \times R_2$ or $(b, 1) \in P_1 \times R_2$. Therefore, either $a \in P_1$ or $b \in P_1$ and P_1 is almost gr-prime.

Similarly, if $A = R_1 \times P_2$ for some graded ideal P_2 of R_2, then P_2 is also almost gr-prime. Finally, suppose that $A = I \times J$ for some proper graded ideals I and J of R_1 and R_2 respectively. Let $a \in (I \cap h(R)) - I^2$. Since

$$A - A^2 = (I \times J) - (I \times J)^2 = ((I - I^2) \times J) \cup (I \times (J - J^2)),$$

then $(a, 1)(1, 0) = (a, 0) \in A - A^2$ and so either $(a, 1) \in A$ or $(1, 0) \in A$. Thus, either $1 \in J$ or $1 \in I$ and this contradicts that I and J are proper. Therefore, $I = I^2$ is idempotent in R_1. Similarly, J is idempotent in R_2.

$\Leftarrow$): Suppose that $A = P_1 \times R_2$, where P_1 is an almost gr-prime ideal in R_1. Let $(r_1, r_2)(t_1, t_2) \in (P_1 \times R_2) - (P_1 \times R_2)^2$. Then $(r_1t_1, r_2t_2) \in (P_1 - P_1^2) \times R_2$ and so $r_1t_1 \in P_1 - P_1^2$. Since P_1 is almost gr-prime in R_1 and $r_1t_1 \in h(R)$, then either $r_1 \in P_1$ or $t_1 \in P_1$ and therefore, either $(r_1, r_2) \in P_1 \times R_2$ or $(t_1, t_2) \in P_1 \times R_2$. Similarly, if $A = R_1 \times P_2$, where P_2 is almost gr-prime in R_2, then A is almost gr-prime in $R_1 \times R_2$. Now, suppose that $A = I \times J$, where I and J are idempotent graded ideals in R_1 and R_2 respectively. Then A^2 is idempotent and so A is almost gr-prime in $R_1 \times R_2$.

Recall that an ideal I of a graded ring R is called gr-principal if $I = \langle a \rangle$ for some $a \in h(R)$. Recall also that If I and J are two graded ideals in a graded ring R, then the ideal $(J : I) = \{x \in R : xI \subseteq J\}$ is a graded ideal, see [6]. In the following lemma, we can justify a condition under which a gr-principal ideal is gr-prime in a graded ring.

Lemma 2.4. *Let R be a graded ring and let $0 \neq a \in h(R)$ be non unit in R. If $\langle a \rangle$ is almost gr-prime with $(0 : \langle a \rangle) \subseteq \langle a \rangle$, then $\langle a \rangle$ is a gr-prime ideal of R.*

Proof. Suppose $\langle a \rangle$ is not gr-prime. Then there exist $x, y \in h(R)$ such that $xy \in \langle a \rangle$ but $x \notin \langle a \rangle$ and $y \notin \langle a \rangle$. If $xy \notin \langle a^2 \rangle$, then $x \in \langle a \rangle$ or $y \in \langle a \rangle$ since $\langle a \rangle$ is almost gr-prime, which is a contradiction. Thus,

$xy \in \langle a^2 \rangle$ and so $x(y+a) \in \langle a \rangle$. If $x(y+a) \notin \langle a^2 \rangle$, then we have $x \in \langle a \rangle$ or $y + a \in \langle a \rangle$. Again in both cases we have a contradiction. Therefore, $x(y+a) \in \langle a^2 \rangle$ and since $xy \in \langle a^2 \rangle$, we get $xa \in \langle a^2 \rangle$. Hence, $xa = ra^2$ for some $r \in R$ and, then $a(x - ar) = 0$. Since $(0 : \langle a \rangle) \subseteq \langle a \rangle$, we get $x - ar \in \langle a \rangle$ and so $x \in \langle a \rangle$, a contradiction. It follows that $\langle a \rangle$ is a gr-prime ideal of R.

Recall that if I is a graded ideal of a graded ring R, then an element $a \in h(R)$ is called a zero divisor on R / I if there is $b \in h(R) - I$ such that $ab \in I$. Recall also that a graded ideal I of a graded ring R is called gr-invertible if there is a graded ideal J of R (denoted by I^{-1}) with $IJ = R$.

Proposition 2.5. *Let P be an n-almost gr-prime ideal in a graded ring R.*

1. *If $a \in h(R)$ is a zero divisor on R / P, then either $a \in P$ or $aP \subseteq P^n$.*

2. *If J is a graded ideal consists of zero divisors on R / P and $J \subseteq P$, then $JP^{n-1} = P^n$.*

3. *If P is gr-invertible, then P is gr-prime.*

Proof. 1. Since a is a zero divisor on R / P, then there is $b \in h(R) - P$ such that $ab \in P$. Suppose that $a \notin P$. Since P is an n-almost gr-prime ideal, then $ab \in P^n$. Also, for any $x \in P \cap h(R)$, we have $x + b \notin P$ and $a(x+b) \in P$. Therefore, $a(x+b) \in P^n$, since P is n-almost gr-prime and so $ax \in P^n$ as $ab \in P^n$. It follows that $aP \subseteq P^n$. Indeed, if $z \in P$, then $z = \sum_{i=1}^{k} x_i$, where $x_i \in h(R) \cap P$ for $i = 1, 2, \ldots, k$ and, then $az = \sum_{i=1}^{k} ax_i \in P^n$.

2. Let $a \in J$ and $b \in P^{n-1}$. It is enough to prove that $ab \in P^n$. Since a is a zero divisor on R / P, then by (1) either $a \in P$ or $aP \subseteq P^n$. If $a \in P$, then we are done. If $aP \subseteq P^n$, then $ab \in aP^{n-1} \subseteq aP \subseteq P^n$.

3. Suppose that there exist $x, y \in h(R)$ such that $xy \in P$ but $x \notin P$ and $y \notin P$. Then y is a zero divisor on R / P and so $yP \subseteq P^n$. Since P is gr-invertible, then P^{-1} exists and $yPP^{-1} \subseteq P^n P^{-1}$ and so $yR \subseteq P^{n-1}$. Therefore, $y \in P^{n-1} \subseteq P$, a contradiction. So, P is a gr-prime ideal in R.

If R is a graded ring and $a \in h(R)$, then similar to the non graded case, one can apply Zorn's lemma to prove that a is a unit in R if and only if $a \notin M$ for any gr–maximal ideal M of R.

Proposition 2.6. *Let R be a gr-local ring with unique gr-maximal ideal M and let I be a graded ideal of R with $M^2 \subseteq I \subseteq M$. Then I is almost gr-prime if and only if $M^2 = I^2$.*

Proof. $(\Rightarrow)$: Suppose that I is almost gr-prime. Let $x, y \in M \cap h(R)$. We prove that $xy \in I^2$. Suppose that $xy \notin I^2$. Since $xy \in M^2 \subseteq I$ and I is almost gr-prime, then either $x \in I$ or $y \in I$. Let $x \in I$, then $y \notin I$ since otherwise $xy \in I^2$. Now, $y^2 \in M^2 \subseteq I$ and so y is a zero divisor on R / I. Hence, by Proposition 2.5, $xy \in yI \subseteq I^2$. Therefore, $(M \cap h(R))^2 \subseteq I^2$ and so clearly $M^2 \subseteq I^2$. The other inclusion is obvious.

$(\Leftarrow)$: Suppose that $M^2 = I^2$. Let $x, y \in h(R)$ with $xy \in I - I^2$. If $x \notin M$, then x is a unit in R and so $y \in I$. Similarly, if $y \notin M$, then $x \in I$. If $x, y \in M$, then $xy \in M^2 = I^2$ which is not true. So, in each case I is an almost gr-prime ideal of R.

A graded ideal I of a graded ring R is called gr-multiplication in R if whenever J is a graded ideal of R with $J \subseteq I$, then there is a graded ideal K of R such that $J = KI$. Clearly, any gr-principal ideal of a graded ring is gr-multiplication. The following lemma can be considered as the graded version of Nakayama's lemma.

Lemma 2.7. *If I is a gr-multiplication ideal of a graded ring R and $A \subseteq J^g(R)$, then $I = IA$ implies that $I = 0$.*

Proof. See [3].

Lemma 2.8. *Let R be a gr-local ring with unique gr-maximal ideal M. If every proper gr-principal ideal in R is almost gr-prime, then $M^2 = 0$.*

Proof. It is enough to prove that $h(M^2) = 0$. Let $x, y \in h(M) - \{0\}$. Consider the gr-principal ideal $\langle xy \rangle$ of R. If $\langle xy \rangle \neq 0$, then by assumption $\langle xy \rangle$ is almost gr-prime with $xy \in \langle xy \rangle$. Thus, either $x \in \langle xy \rangle$, $y \in \langle xy \rangle$ or $xy \in \langle xy \rangle^2$. If $x \in \langle xy \rangle$, then $\langle x \rangle = \langle x \rangle \langle y \rangle$ and so $\langle x \rangle = 0$ by Lemma 2.7. Similarly, if $y \in \langle xy \rangle$, we get $\langle y \rangle = 0$. Finally, if $xy \in \langle xy \rangle^2$, then $\langle xy \rangle = \langle xy \rangle^2$ and so again by Lemma 2.7, $\langle xy \rangle = 0$. Therefore, in any case, we get a contradiction to the assumption that $x, y, xy \notin \{0\}$. Hence, $xy = 0$ and $h(M^2) = 0$.

Lemma 2.9. *Let R be a graded ring in which every proper gr-principal ideal is almost gr-prime. Then for all $a \in h(R)$, we have $\langle a^2 \rangle = \langle a^3 \rangle$. Moreover, $\langle a^2 \rangle = \langle e \rangle$ for some idempotent element $e \in h(R)$.*

Proof. Let M be a gr-maximal ideal of R. By Proposition 2.1, every proper gr-principal ideal of R_M^g is almost gr-prime and so $M^2 R_M^g = 0 R_M^g$ by Lemma 2.8. Let $a \in h(R)$. If $a \in M$, then $\langle \frac{a}{1} \rangle^2 \in M^2 R_M^g$ and so $\langle \frac{a}{1} \rangle^2 = 0 R_M^g$. If $a \notin M$, then $\frac{a}{1}$ is a unit in

R_M^g. In both cases, $\langle \frac{a}{1} \rangle^2 = \langle \frac{a}{1} \rangle^3$ and so $\langle a^2 \rangle R_M^g = \langle a^3 \rangle R_M^g$ for every gr-maximal ideal M of R. Hence, $\langle a^2 \rangle = \langle a^3 \rangle$. For the other part, $\langle a^2 \rangle = \langle a^3 \rangle$ implies that $\langle a^2 \rangle = \langle a^4 \rangle$ and so $a^2 = ba^4$ for some $b \in R$. Thus, $e = ba^2 = b^2a^4 = (ba^2)^2 = e^2$ is idempotent. Moreover, $ea^2 = ba^4 = a^2$ and then $\langle a^2 \rangle \subseteq \langle e \rangle \subseteq \langle a^2 \rangle$. It follows that $\langle a^2 \rangle = \langle e \rangle$.

Definition 2.10. *A* graded ring R is called gr-regular if for every $a \in h(R)$, there exists $x \in R$ satisfying $a = axa$.

Recall that a graded ring R is called gr-field if every nonzero homogeneous element in R is a unit. Eqivalentely, R is gr-field if and only if R has no proper graded ideals if and only if 0 is a gr-maximal ideal in R.

Similar to the non graded case, we can see the following characterization of gr-regular rings.

Lemma 2.11. *The following are equivalent for a graded ring R.*

1. *R is a gr-regular ring.*

2. *R_M^g is a gr-field for each gr-maximal ideal M of R.*

3. *Every graded ideal of R is idempotent.*

Proof. $(1. \Rightarrow 2.)$: Let M be a gr-maximal ideal of R. We prove that $MR_M^g = 0R_M^g$. Let $a \in M \cap h(R)$. Then there exists $x \in R$ such that $a = axa$ and so $e = ax = axax = (ax)^2 = e^2$ is an idempotent element in M. Therefore, $e(e-1) = 0$ and $e - 1 \notin M$. Hence, $\frac{e}{1} = \frac{0}{1}$ and then $\frac{a}{1} = \frac{ea}{1} = \frac{e}{1}\frac{a}{1} = \frac{0}{1}$. Thus, $(M \cap h(R))R_M^g = 0R_M^g$ and then $MR_M^g = 0R_M^g$.

$(2. \Rightarrow 3.)$: Let I be a graded ideal in R. Let M be a gr-maximal ideal in R. Then $IR_M^g = 0R_M^g$ or $IR_M^g = R_M^g$ since R_M^g is a field. Therefore, $I^2 R_M^g = IR_M^g$ for each gr-maximal ideal M of R and then $I^2 = I$.

$(3. \Rightarrow 1.)$: Let $a \in h(R)$. Then $\langle a \rangle = \langle a^2 \rangle$ and so $a = a^2 r$ for some $r \in R$. Hence, $a = ara$ and R is gr-regular.

Remark 2.12. If $R = \bigoplus_{n \in Z} R_n$ is a $\mathbb{Z}$-graded ring and R is a field, then by [5], R is concentrated in R_0. This means that $R = R_0$ and $R_n = 0$ for all $0 \neq n \in Z$. Moreover, If M is gr-maximal ideal in R, then one can see that

$$M = \cdots \oplus R_{-2} \oplus R_{-1} \oplus M_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

for some maximal ideal M_0 of R_0, see [5]. Therefore, in this case R is a gr-regular if and only if R_M^g is a field for each gr-maximal ideal M of R.

Recall that if R is gr-indecomposable, then 0 and 1 are the only idempotent elements in R. Indeed, if e is idempotent in R, then $R \cong \langle e \rangle \times \langle 1 - e \rangle$. In the next main Theorem, we give a characterization of graded rings in which every proper graded ideal is almost gr-prime. First, we need the following definition and lemma.

Definition 2.13. Let R be a graded ring. The gr-nilradical of R (denoted by $nil^g(R)$) is defined as

$$nil^g(R) = \{x \in h(R) : x^n = 0 \textit{ for some } n \in \mathbb{N}\}.$$

Similar to the non graded case one can prove that

$$nil^g(R) = \bigcap \{P : P \textit{ is gr-prime in } R\}.$$

Lemma 2.14. *Let R be a graded ring. If M is the set of homogeneous nonunit element in R which is contained in $nil^g(R)$, then M is a graded ideal of R. Moreover, R is gr-local with M as the unique gr-maximal ideal.*

Proof. Since $M \subseteq nil^g(R)$ and 1 is a unit, then clearly M is a proper ideal of R. Moreover, M is graded since $M \subseteq h(R)$. Since $0 \in M$, then 0 is nonunit of R and so $0 \neq 1$. Thus R is not trivial and so it has at least one gr-maximal ideal. Let I be one such. Then I contains only nonunit elements and so, $I \subseteq M \subset R$. Hence, $I = M$ since I is gr-maximal. Therefore, M is the unique gr-maximal ideal of R and R is gr-local.

Theorem 2.15. *Let R be a graded ring. The following are equivalent.*

1. *Every proper graded ideal of R is almost gr-prime.*

2. *Every proper gr-principal ideal of R is almost gr-prime.*

3. *R is either gr-regular or gr-local with $M^2 = 0$, where M is the gr-maximal ideal of R.*

Proof. $(1. \Rightarrow 2.)$: Trivial.

$(2. \Rightarrow 3.)$: We first assume that R is gr-indecomposable. Then by Lemma 2.9, $\langle a^2 \rangle = \langle e \rangle$ for some idempotent e in R and so $a^2 = 0$ or $a^2 = 1$. If a is a unit, then $a^2 = 1$. Thus, $a^2 = 0$ for any nonunit a in R and so the set of homogeneous nonunits of R is contained in $nil^g(R)$ and so it is a graded ideal of R and R is gr-local by Lemma 2.14. Therefore, $M^2 = 0$ by Lemma 2.8. Next we assume that R is gr-decomposable, say, $R = R_1 \times R_2$. We prove that R is gr-regular. Clearly, if R_1 and R_2 are gr-regular, then R is so. Suppose that R_2 is not gr-regular. Then by Lemma 2.11, it has a non idempotent graded ideal I. Consider the graded ideal $0 \times I$ of R. Let $r \in I - I^2$. Then $(1, r)(0, 1) = (0, r) \in 0 \times I - (0 \times I)^2$ and so either $(1, r) \in 0 \times I$ or $(0, 1) \in 0 \times I$ since $0 \times I$ is almost gr-prime in R. But, neither $(1, r)$ nor $(0, 1)$ belong to $0 \times I$ and we get a contradiction. Therefore, R_2 is gr-regular. Similarly, we can see that R_1 is gr-regular and hence R is gr-regular.

$(3. \Rightarrow 1.)$: Suppose that R is gr-local with $M^2 = 0$. Let I be a proper graded ideal of R. Let $a, b \in h(R)$ such that $ab \in I - \{0\}$. Since $M^2 = 0$ and $ab \neq 0$, then either $a \notin M$ or $b \notin M$. If $a \notin M$, then a is a unit and so $b = a^{-1}(ab) \in I$. Similarly, if $b \notin M$, then $a \in I$. Therefore, I is an almost gr-prime ideal of R. On the other hand, if R is gr-regular, then any proper graded ideal in R is idempotent by Lemma 2.11 and hence it is almost gr-prime.

3. Further Properties of Almost GR-Prime Ideals

In this section we introduce some definitions, and then we determine four properties of almost gr-prime ideals.

Definition 3.1. Let $R = \underset{g \in G}{\oplus} R_g$ be a G-graded ring, where G is an abelian group. Then R is said to be Noetherian graded ring if the graded ideals in R satisfy the ascending chain condition.

Definition 3.2. A commutative G-graded ring R is said to be valuation graded ring if for any $a, b \in h(R)$ either a divides b or b divides a, that is either $b = ca$ for some $c \in h(R)$ or $a = db$ for some $d \in h(R)$.

Definition 3.3. A commutative G-graded ring R is called a graded domain if $1 \in R_e$, where e is the identity element in G, and whenever $a \in h(R)$ with $a \neq 0$, then $ab = 0$ if and only if $b = 0$.

Lemma 3.4. *Let R be a graded domain and let $c \neq 0 \in h(R)$ such that c is nonunit. If $< c >$ is not graded prime ideal, then there exists $a, b \in h(R) - h(< c >)$ with $ab \in < c >$, but $ab \notin < c^n >$.*

Proof. Let $a, b \in h(R)$ with $ab \in < c >$ and $a \notin < c >$ and $b \notin < c >$. If $ab \notin < c^n >$, we are done. So assume that $ab \in < c^n >$; then $a(b + c^{n-1}) \in < c >$ and $a, b + c^{n-1} \notin h(< c >)$. If $a(b + c^{n-1}) \in < c^n >$, then $ac^{n-1} \in < c^n >$, and this implies that $ac^{n-1} = dc^n$ for some

$d \in h(R)$, so $ac^{n-1} - dc^{n} = 0 \Rightarrow (a - dc)c^{n-1} = 0 \Rightarrow a = dc \in < c >$, contradiction.

Corollary 3.5. If $< c >$ is an n-almost graded prime ideal in a graded domain R, then $< c >$ is a graded prime ideal.

Lemma 3.6. (*Generalized Nakayama*) *If J is a graded ideal in a commutative graded ring R with identity, then the following conditions are equivalent*

(1) *J is contained in every maximal graded ideal in R.*

(2) $1 - j_e$ *is a unit for every* $j_e \in J_e$.

(3) *If A is a finitely generated graded G-module over R such that* $JA = A$, *then* $A = 0$.

(4) *If B is a graded submodule of a finitely generated graded G-module A over R such that* $A = JA + B$, *then* $A = B$.

Proof. $(1) \Rightarrow (2)$ If $j_e \in J_e$ such that $1 - j_e$ is not a unit, then the graded ideal $< 1 - j_e >$ is not R itself and by Zorn's Lemma is contained in a maximal graded ideal $M \neq R$. But $1 - j_e \in M$ and $j_e \in M$ which implies that $1 \in M$, which is a contradiction. Therefore $1 - j_e$ is a unit.

$(2) \Rightarrow (3)$ Since A is finitely generated, there must be a minimal generating set $X = \{a_1, a_2, \ldots, a_n\} \subseteq h(A)$. If $A \neq 0$, then $a_1 \neq 0$ by minimality. Since $JA = A$,

$$a_1 = j_e a_1 + j_2 a_2 + \ldots + j_n a_n,$$

where $j_e \in J_e$ and $j_i \in h(J)$ for all $i = 2, \ldots, n$, whence $1a_1 = a_1$ so that $(1 - j_e)a_1 = 0$ if $n = 1$ and

$$(1 - j_e)a_1 = j_2 a_2 + \ldots + j_n a_n \quad \text{if } n > 1.$$

Since $1 - j_e$ is a unit in R, thus if $n = 1$, then $a_1 = 0$ which is a contradiction. If $n > 1$, then a_1 is a linear combination of $a_2, \ldots, a_n$.

Consequently, $\{a_2, \ldots, a_n\}$ generates A, which contradicts the choice of X. Therefore $A = 0$.

$(3) \Rightarrow (4)$ Consider the quotient graded module A / B, then one can easily check that $J(A / B) = A / B$ whence $A / B = 0$ and $A = B$ by (3).

$(4) \Rightarrow (1)$ Let M be any maximal graded ideal. The graded ideal $J + M = JR + M$ contains M. But $JR + M \neq R$ (otherwise $R = M$ by (4)). Consequently, $JR + M = M$ by maximality. Therefore $J = JR \subseteq M$.

Corollary 3.7. Let R be a Noetherian commutative graded ring with identity, and let I be a graded ideal in R such that I is contained in every maximal graded ideal in R, then $\bigcap_{n=1}^{\infty} I^n = \{0\}$.

Proof. Let $A = \bigcap_{n=1}^{\infty} I^n$, then A finitely generated graded module over R, since R is Noetherian and $IA = A$, then by Lemma 3.6 $A = 0$.

In a Noetherian graded domain R we have the following result.

Theorem 3.8. *Let R be a Noetherian graded domain, and let I be a graded ideal in R such that I is contained in every maximal graded ideal in R. Then I is a graded prime ideal if and only if I is an n-almost graded prime ideal for all $n \geq 2$.*

Proof. Let I be an n-almost graded prime ideal for all $n \geq 2$. Let $x, y \in h(I)$. If $xy \notin I^n$ for some n in $\mathbb{Z}^+$, then $xy \in h(I) - h(I^n)$. Hence $x \in h(I)$ or $y \in h(I)$, since I is an n-almost graded prime ideal. If $xy \in I^n$ for all $n \geq 1$, then $xy \in J = \bigcap_{n=1}^{\infty} I^n$. Since R is a Noetherian graded domain, then by Corollary 3.7, $J = \{0\}$. Thus $xy = 0$ implies $x = 0 \in I$ or $y = 0 \in I$, since R is a graded domain.

Conversely, if I is a graded prime ideal, then one can easily check that I is an n-almost graded prime ideal for all $n \geq 2$.

Definition 3.9. A commutative graded ring R is said to be valuation graded ring, if for any $a, b \in h(R)$ either a divides b or b divides a.

In a valuation graded domain V, we have the following result about almost graded prime ideals.

Theorem 3.10. *Let V be a valuation graded domain. Then a graded ideal I of V is almost graded prime if and only if it is a graded prime ideal.*

Proof. Let $x, y \in h(V)$ with $xy \in I$. Assume that $x, y \notin h(I)$. Since $x \notin I, x$ divides a for all $a \in h(I)$, which implies that $I \subseteq < x >$, similarly $I \subseteq < y >$. Thus $< xy > \supseteq I^2$. If $I^2 \neq < xy >$, then $xy \in h(I) - h(I^2)$. Since I is almost, $x \in I$ or $y \in I$, a contradiction.

So assume $< xy > = I^2$. Then I being a factor of a graded principle ideal is graded principle. Thus by Corollary 3.5, I is a graded prime ideal. The converse is trivial for all graded rings R.

Now we have the following main result for the local graded ring (R, M).

Theorem 3.11. *Let (R, M) be a local graded ring. Every proper graded ideal of R is a product of almost graded prime ideals if and only if either M is principle or*

(i) *for each $x \in h(M) - h(M^2)$, $< x^2 > = M^2$; and*

(ii) $M^3 = \{0\}$.

Proof. $(\Rightarrow)$ Suppose M is not principle. Let I be a proper graded ideal of R which is a product of almost graded prime ideals with $M^2 \subset I \subset M$. Note that I is actually almost graded prime since I can not be a product of two proper graded ideals. Then by Proposition 2.6, $I^2 = M^2$. Now for $x \in h(M) - h(M^2)$, take $I = < x > + M^2$, so $M^2 \subset I \subseteq M$. Then

$$M^2 = (<x> + M^2)^2 \subseteq <x^2> + <x> M^2 + M^4$$
$$\subseteq <x^2> + M^3$$
$$\subseteq <x^2> + MM^2 \subseteq M^2.$$

Thus $M^2 =< x^2 >$. Now we show that $M^3 = \{0\}$. Consider $(0 : x)$. We have $M^2 \subseteq< x >$. If $(0 : x) \subseteq M^2$, then $(0 : x) \subseteq< x >$ where $< x >$ is almost graded prime ideal. By Lemma 2.4, $< x >$ is graded prime ideal which implies that $< x >= M$, a contradiction. So assume that $(0 : x) \not\subseteq M^2$. Let $y \in h((0 : x)) - h(M^2)$. Since $(0 : x) \subseteq M$, then $M^2 =< y^2 > \subset< y >\subset (0 : x)$. Thus $x^2 \in M^2 \subseteq (0 : x)$; so $x^3 = 0$ and hence $xM^2 = x < x^2 >= \{0\}$. Now let $< z >\subseteq M^3$. Then $< z >\subseteq M < x^2 >$, but $M < x^2 >\subseteq xM^2 = \{0\}$. Hence $M^3 = \{0\}$.

($\Leftarrow$) If M is principle, say $M =< m >$, for some $m \in h(R)$, then the only proper graded ideals of R are $< m >, < m^2 >, \cdots, < m^k >, \cdots$ each of which is a product of graded prime ideals.

Now assume that M is not principle and suppose (i) and (ii) hold. Then by (i), M^2 is principle and hence no proper graded ideal between $\{0\}$ and M^2, because if $I \subseteq M^2$, then $I \subseteq MM^2 = M^3 = \{0\}$, since M^2 is a gr-multiplication graded ideal in R. So suppose $I \subseteq M$ is a graded proper ideal with $I \not\subseteq M^2$. Let $x \in h(I) - h(M^2)$, then $I \supseteq< x > \supseteq< x^2 >= M^2$.

If $I = M^2$, then I is a product of graded prime ideals. So assume that $I \supset M^2$, we have $< x >\subset I$, so $M^2 =< x^2 >\subseteq I^2$, but since $I \subseteq M$, we have $I^2 \subseteq M^2$, hence $I^2 = M^2$. So by Proposition 2.6, I is almost graded prime ideal.

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